S.T. Yau College Student Mathematics Contests 2019 Ma
Algebra and Number Theory Overall

1. This problem is meant to show for any prime number p, that there exists an irreducible polynomial of degree p with $(p-2)$ real roots and 2 (nonreal) complex conjugate roots. Let

$$P_0(X) = \prod_{k=1}^{p-2} (X+k)\,(X^2+1)$$

(a) Prove that for any integer k, that $P_k(X) = kp^2 P_0(X) + X^p - p$ is an irreducible polynomial in $\mathbb{Z}[X]$

(b) Deduce from P_k a sequence of polynomials in $\mathbb{Q}[X]$ converging uniformly to P_0 on any compact subset of $\mathbb{C}$.

(c) Prove that for k large enough, P_k has two complex conjugates roots and $(p-2)$ real roots.

2. Let k be a field and take $L = k(x, y)$, where x is transcendental over k and $x^2 + y^2 = 1$. Find the Galois group of L over k.

3. Let G be a finite group and $\rho : G \longrightarrow GL(V)$ a representation on a complex finite dimensional vector space V. Let $\rho^* : G \longrightarrow GL(V^*)$ be the dual representation.

Consider the symmetric algebra $S(V)$, viewed as polynomial functions on V^* For $l \in V^*$ and $p \in S(V)$, denote by p_l the fonction on G given by: $p_l(g) = p(g(l))$; this defines a function

$$\begin{array}{ccc} S(V) & \xrightarrow{\phi_l} & \mathbb{C}[G] \\ p & \longrightarrow & p_l \end{array}$$

(a) Prove $\exists\, u \in V^*$ whose stabilizer is the neutral element of G.

(b) Prove that ϕ_u is surjective.

(c) Prove that for any faithful irreducible representation ρ of G, there exists an integer n such that ρ is equivalent to a subrepresentation de $S^n(V)$.